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NUMERICAL ANALYSIS OF CONVECTION–DIFFUSION PROBLEMS USING THE BOUNDARY ELEMENT METHOD

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ABSTRACT

This paper presents a boundary element formulation for transient convection–diffusion problems employing the fundamental solution of the corresponding steady-state equation with constant coefficients and a dual reciprocity approximation. The formulation allows the mathematical problem to be described in terms of boundary values only. Numerical results show that the BEM does not present oscillations or damping of the wave front as appear in other numerical techniques.

KEY WORDS Boundary element method Convection–diffusion problems

INTRODUCTION

The solution of the convection–diffusion equation is a challenging task for all numerical methods because of the nature of the equation, which includes first-order and second-order partial derivatives in space. According to the value of the Péclet number, the equation becomes parabolic (for diffusion-dominated processes) or hyperbolic (for convection–dominated processes). Traditional finite difference and finite element algorithms are generally accurate for solving the former but not the latter, in which case oscillations and smoothing of the wave front are introduced. This can be interpreted as an 'artificial diffusion' intrinsic to these methods[1–4].

Applications of the boundary element method to steady-state convection–diffusion have shown that the BEM seems to be relatively free from these problems[5–7]. This was also the case for some transient applications using formulations with time-dependent fundamental solutions[8,9]. The main restriction of these formulations, however, is the fact that fundamental solutions are only available for equations with constant coefficients, or coefficients with very simple variations in space[10]. Alternative formulations for transient problems have employed the fundamental solution of the diffusion equation and treated the convective terms as pseudo-sources, thus allowing for variable velocity fields[11]. The disadvantage of such approach is that a domain discretization is required to account for the pseudo-sources.

This work presents a boundary element formulation for the solution of transient convection–diffusion problems based on a dual reciprocity scheme[12]. The domain integral resulting from the dual reciprocity approach is transformed into equivalent boundary integrals by expanding the time-derivative term as a summation of approximating functions and introducing particular solutions which satisfy an associated non-homogeneous steady-state convection–diffusion equation. Thus, the problem is ultimately described in terms of boundary values only, consequently reducing its dimensionality by one. Although only problems with constant velocity are analysed herein, the formulation can be extended to deal with variable velocity fields using a similar dual reciprocity approach[13].

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Results of several analyses are presented and compared to analytical solutions. They show that the boundary element formulation developed in this work does not display any artificial diffusion or oscillatory behaviour, thus precluding the need for 'upwind' or other algorithms common to finite element analysis.

BEM FORMULATION FOR STEADY-STATE PROBLEMS

The two-dimensional steady-state convection–diffusion equation including first-order reaction can be written in the form:

$$D\nabla^2\phi - v_x\frac{\partial\phi}{\partial x} - v_y\frac{\partial\phi}{\partial y} - k\phi = 0 \tag{1}$$

where v_x and v_y are the components of the velocity vector $\mathbf{v}$, D is the diffusivity coefficient (assuming the medium is homogeneous and isotropic) and k represents the reaction coefficient. The variable ϕ can be interpreted as temperature for heat transfer problems, concentration for dispersion problems, etc. and will be herein referred to as a potential. The mathematical description of the problem is complemented by boundary conditions of the Dirichlet, Neumann or Robin (mixed) types.

The above differential equation can be transformed into an equivalent integral equation by applying a weighted residual technique. Starting with the weighted residual statement:

$$\int_\Omega \left(D\nabla^2\phi - v_x\frac{\partial\phi}{\partial x} - v_y\frac{\partial\phi}{\partial y} - k\phi\right)\phi^* \,\mathrm{d}\Omega = 0 \tag{2}$$

and integrating by parts twice the Laplacian and once the first-order derivatives the following equation is obtained:

$$\phi(\xi) = D\int_\Gamma \phi^*\frac{\partial\phi}{\partial n}\mathrm{d}\Gamma - D\int_\Gamma \phi\frac{\partial\phi^*}{\partial n}\mathrm{d}\Gamma - \int_\Gamma \phi\phi^* v_n \,\mathrm{d}\Gamma \tag{3}$$

where $v_n = \mathbf{v}\cdot\mathbf{n}$, $\mathbf{n}$ is the unit outward normal vector and the dot stands for scalar product.

In the above equation, ϕ^* is the fundamental solution of (1), i.e. the solution of:

$$D\nabla^2\phi^* + v_x\frac{\partial\phi^*}{\partial x} + v_y\frac{\partial\phi^*}{\partial y} - k\phi^* = -\delta(\xi,\chi) \tag{4}$$

in which ξ and χ are the source and field points, respectively, and δ is the Dirac delta. It can be noticed that the sign of the first-order derivatives is reversed in (1) and (4), since this operator is not self-adjoint. For two-dimensional problems, ϕ^* is of the form:

$$\phi^*(\xi,\chi) = \frac{1}{2\pi D}\mathrm{e}^{-\mathbf{v}\cdot\mathbf{r}/2D}K_0(\mu r) \tag{5}$$

where

$$\mu = \left[\left(\frac{|\mathbf{v}|}{2D}\right)^2 + \frac{k}{D}\right]^{\frac{1}{2}} \tag{6}$$

and r is the modulus of $\mathbf{r}$, the distance vector between the source and field points. The derivative of the fundamental solution with respect to the outward normal direction is given by:

$$\frac{\partial\phi^*}{\partial n} = \frac{1}{2\pi D}\mathrm{e}^{-\mathbf{v}\cdot\mathbf{r}/2D}\left[-\mu K_1(\mu r)\frac{\partial r}{\partial n} - \frac{v_n}{2D}K_0(\mu r)\right] \tag{7}$$

In the above, K_0 and K_1 are Bessel functions of second kind, of orders zero and one, respectively.

Equation (3) permits calculating the value of ϕ at any internal point once the boundary values of ϕ and $\partial\phi/\partial n$ are all known. In order to obtain a boundary integral equation, the source point ξ is taken to the boundary and a limit analysis carried out due to the jump of $\partial\phi^*/\partial n$. The result is:

$$c(\xi)\phi(\xi)=D\int_\Gamma \phi^*\frac{\partial\phi}{\partial n}\mathrm{d}\Gamma-D\int_\Gamma \phi\frac{\partial\phi^*}{\partial n}\mathrm{d}\Gamma-\int_\Gamma \phi\phi^* v_n\,\mathrm{d}\Gamma \tag{8}$$

in which $c(\xi)$ is a function of the internal angle the boundary Γ makes at point ξ[14].

NUMERICAL SOLUTION

For the numerical solution of the problem, (8) is written in a discretized form in which the integrals over the boundary are approximated by a summation of integrals over individual boundary elements, i.e.

$$c_i\phi_i=D\sum_{j=1}^{N}\int_{\Gamma_j}\phi^*\frac{\partial\phi}{\partial n}\mathrm{d}\Gamma-D\sum_{j=1}^{N}\int_{\Gamma_j}\left(\frac{\partial\phi^*}{\partial n}+\frac{v_n}{D}\phi^*\right)\phi\,\mathrm{d}\Gamma \tag{9}$$

where the index i stands for values at the source point ξ and N elements have been employed. In the above equation, it can be seen that:

$$\frac{\partial\phi^*}{\partial n}+\frac{v_n}{D}\phi^*=\frac{1}{2\pi D}\mathrm{e}^{-\mathbf{v}\cdot\mathbf{r}/2D}\left[-\mu K_1(\mu r)\frac{\partial r}{\partial n}+\frac{v_n}{2D}K_0(\mu r)\right] \tag{10}$$

Next, the variation of functions ϕ and $\partial\phi/\partial n$ within each element are approximated by interpolating from the values at the element nodes. Herein, linear elements are used, for which the expressions are:

$$\phi=\Phi_1\phi_1+\Phi_2\phi_2$$

$$\frac{\partial\phi}{\partial n}=q=\Phi_1 q_1+\Phi_2 q_2$$

where Φ_1 and Φ_2 are linear interpolation functions. Substituting the above into (9), the following expression is obtained:

$$c_i\phi_i=\sum_{j=1}^{N}(g_{ij}^1 q_j^1+g_{ij}^2 q_j^2-h_{ij}^1\phi_j^1-h_{ij}^2\phi_j^2) \tag{11}$$

Note that the indexes 1 and 2 refer to the nodal (extreme) points of each element, and

$$g_{ij}^k=D\int_{\Gamma_j}\Phi_k\phi^*\,\mathrm{d}\Gamma$$

$$h_{ij}^k=D\int_{\Gamma_j}\Phi_k\left(\frac{\partial\phi^*}{\partial n}+\frac{v_n}{D}\phi^*\right)\mathrm{d}\Gamma$$

Adding up the contributions of adjoining elements to each nodal point, (11) can be rewritten as:

$$c_i\phi_i=\sum_{j=1}^{N}(G_{ij}q_j-H_{ij}\phi_j) \tag{12}$$

The above equation involves N values of ϕ and N values of q, half of which are prescribed as boundary conditions. In order to calculate the remaining unknowns, it is necessary to generate N equations. This can be done by using a simple collocation technique, i.e. by making the equation be satisfied at the N nodal points. The result is a system of equations of the form:

$$\mathbf{H}\boldsymbol{\phi}=\mathbf{G}\mathbf{q} \tag{13}$$

where the c_i values have been incorporated into the diagonal coefficients of matrix **H**. After introducing the boundary conditions, the system is reordered and solved by a direct method, e.g. Gauss elimination.

Evaluation of the coefficients of matrices **H** and **G** is carried out numerically. For the off-diagonal terms, a selective Gaussian integration with number of integration points as a function of the distance between source point and field element is employed[14]. The diagonal coefficients of matrix **G** have a weak singularity of the logarithmic type, and are calculated using the self-adaptive scheme of Telles[15]. The coefficients H_{ii} can be calculated, in the absence of the reaction term, by noting that a consistent solution for a prescribed uniform potential along the boundary can only be obtained if matrix **H** is singular, i.e.

$$H_{ii} = -\sum_{j=1}^{N} H_{ij} \quad (i \neq j) \tag{14}$$

However, when $k \neq 0$, there is flux when a uniform potential is applied (or, in other words, the zero flux state is achieved for non-uniform potential distribution). In this case, the coefficients H_{ii} have to be evaluated explicitly. These terms are composed of two parts, one being a sum of integrals of the form h_{ij}^k and the other the free term c_i. The former also possesses a logarithmic singularity, and is calculated using Telles' scheme[15]. The free terms c_i depend solely on geometry, and have the same values as for Laplace's equation[13].

APPLICATIONS FOR STEADY PROBLEMS

To show the performance of the boundary element scheme, two numerical applications were studied. The first is the one-dimensional problem of a long bar moving at a constant velocity; the second is a plate with a sinusoidal potential distribution along one face, with first-order reaction.

Moving bar

The BEM formulation was initially tested with the problem of a bar moving parallel to the x-axis with constant velocity v_x and with specified potential at the edges, i.e. $\phi = 300$ at $x = 0$ and $\phi = 0$ at $x = L$. Other values adopted were $D = 1$, $k = 0$. The problem was analysed as two-dimensional with cross-section 6.0×0.7, with the boundary condition $\partial\phi/\partial n = 0$ specified along the faces parallel to the x-axis. Symmetry was taken into account by reflection and condensation[14], thus only the upper half of the region needed be considered. A sketch of the problem is shown in *Figure 1*.

The discretization employed 17 elements on the longer side and 1 element on each of the smaller ones, making up a total of 19 linear elements and 22 nodes, for double nodes were used in the corners to allow for the discontinuity of the normal vector at these points[14].

Results are plotted in *Figures 2* and *3* for several velocity values, compared with the analytical solution:

$$\phi = 300\, e^{(v_x/2)x} \frac{\sinh[(v_x/2)(L-x)]}{\sinh[(v_x/2)L]}$$

It can be seen from the Figures that the agreement between the two solutions is excellent.

To further test the numerical formulation for high velocity values, *Table 1* presents the BEM results obtained with the same discretization of *Figure 1* for a range of values of v_x for which

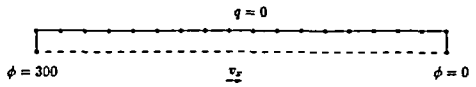

Figure 1 Moving bar: geometry, discretization and boundary conditions

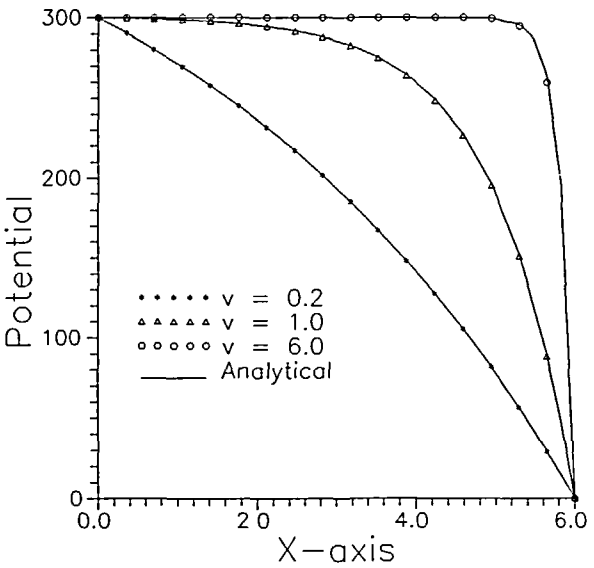

Figure 2 Results for positive velocity

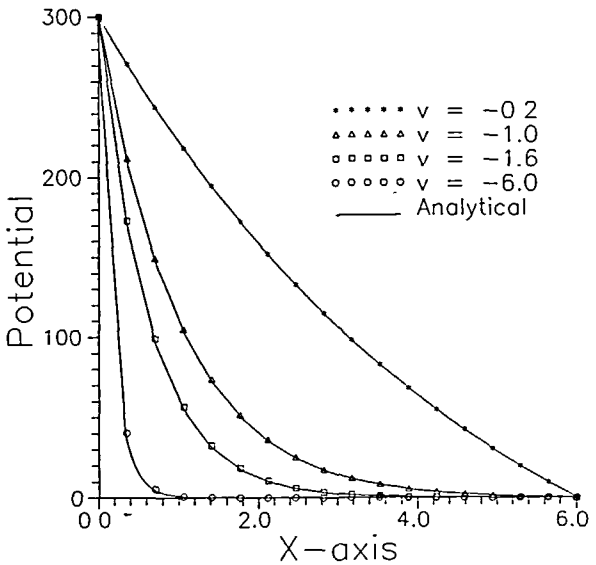

Figure 3 Results for negative velocity

the analytical solution gives $\phi = 300$ at all points in the Table. It is interesting to note that even for $v_x = 500$, corresponding to a local Péclet number of $Pe = 176$, the maximum deviation between the BEM and analytical solutions is less than 10%. It is also of remark that oscillations are more pronounced at the centre of the bar rather than near the boundary layer, as would be expected, and no smoothing of the wave front is introduced.

Table 1 Potential values for different velocities

x	$v_x=50$	$v_x=100$	$v_x=200$	$v_x=300$	$v_x=400$	$v_x=500$
0.3	300.0	300.0	300.3	300.2	298.9	296.7
0.6	300.0	300.0	298.6	296.0	295.4	298.1
1.2	300.0	300.0	300.0	299.7	298.7	297.4
1.8	300.0	300.0	300.0	300.0	299.9	299.7
2.4	300.0	300.1	303.8	311.4	320.9	329.9
3.0	300.0	299.9	301.7	306.4	313.3	321.4
3.6	300.0	299.9	300.8	303.7	308.6	314.7
4.2	300.0	299.9	300.4	302.2	305.6	310.3
4.8	300.0	299.9	300.2	301.3	303.8	307.4
5.4	300.0	299.9	300.1	300.9	302.6	305.4
5.7	300.0	299.9	300.1	300.8	302.2	304.7

Plate with first-order reaction

The potential distribution in a plate with the geometry and boundary conditions shown in *Figure 4* was studied next for a range of values of the velocity v_x (from 0 to 8.33) and the reaction coefficient k (from 0 to 13.88). The discretization employed 20 linear elements and 23 nodes (with double nodes at corners), taking symmetry into account. A unit value was assumed for coefficient D.

The results for the potential along the centre line ($y=l/2$) obtained with the present boundary element scheme are plotted in *Figures 5* to *7*, compared with the following analytical solution:

$$\phi=\frac{e^{m_1(L-x)}-e^{m_2(L-x)}}{e^{m_1L}-e^{m_2L}}\sin\frac{\pi}{l}y$$

where

$$m_i=\frac{1}{2}\left[-v_x\pm\sqrt{v_x^2+4\left(\frac{\pi^2}{L^2}+k\right)}\right](i=1,2)$$

and L and l are the dimensions in the x and y directions, respectively 6.0 and 4.0 in the present case.

It can be seen in the Figures that the results compare very well with the analytical solution for all values of k, showing again no oscillations or damping.

BEM FORMULATION FOR TRANSIENT PROBLEMS

The two-dimensional transient convection–diffusion equation including first-order reaction can be written in the form:

$$D\nabla^2\phi-v_x\frac{\partial\phi}{\partial x}-v_y\frac{\partial\phi}{\partial y}-k\phi=\frac{\partial\phi}{\partial t} \tag{15}$$

where the variables are defined as for the steady-state case. Since the problem is now transient, initial values of ϕ at time t_0 should also be provided.

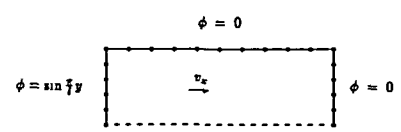

Figure 4 Rectangular plate: geometry, discretization and boundary conditions

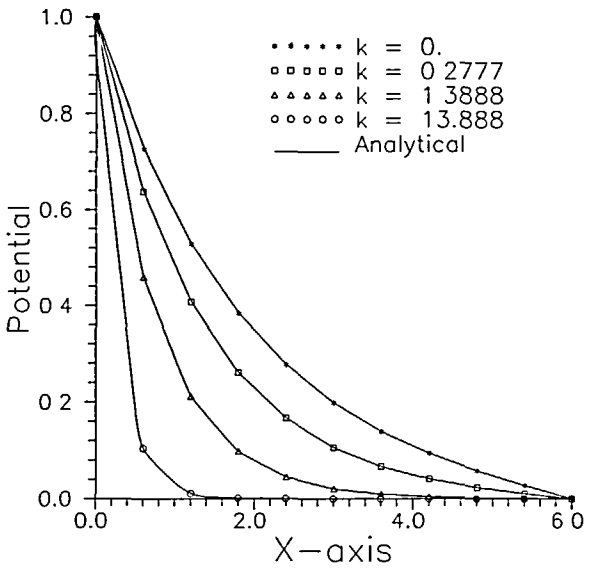

Figure 5 Results for $v_x = 0$

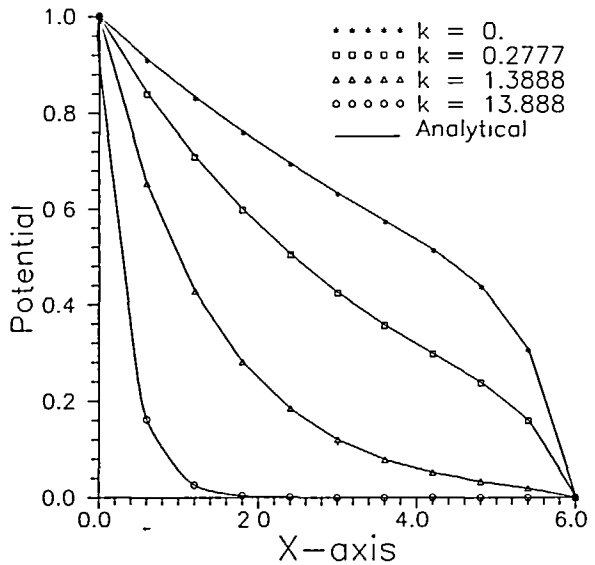

Figure 6 Results for $v_x = 1.66$

Applying a weighted residual technique to the above equation, using as weighting function ϕ^* the fundamental solution of the corresponding steady-state equation (expression 5), we obtain

$$\int_\Omega \left(D\nabla^2\phi - v_x \frac{\partial \phi}{\partial x} - v_y \frac{\partial \phi}{\partial y} - k\phi \right) \phi^* \, d\Omega = \int_\Omega \frac{\partial \phi}{\partial t} \phi^* \, d\Omega \tag{16}$$

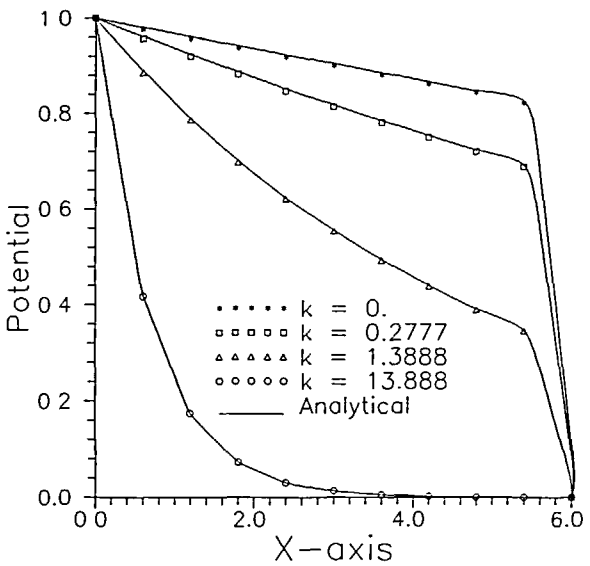

Figure 7 Results for $v_x = 8.33$

Integrating by parts twice the Laplacian and once the first-order space derivatives gives

$$\phi(\xi) - D\int_\Gamma \phi^* \frac{\partial \phi}{\partial n} d\Gamma + D\int_\Gamma \phi \frac{\partial \phi^*}{\partial n} d\Gamma + \int_\Gamma \phi\phi^* v_n d\Gamma = -\int_\Omega \frac{\partial \phi}{\partial t}\phi^* d\Omega \tag{17}$$

The corresponding boundary integral equation, for a source point ξ on the boundary, takes into account the jump of $\partial\phi^*/\partial n$, in the form

$$c(\xi)\phi(\xi) - D\int_\Gamma \phi^* \frac{\partial \phi}{\partial n} d\Gamma + D\int_\Gamma \phi \frac{\partial \phi^*}{\partial n} d\Gamma + \int_\Gamma \phi\phi^* v_n d\Gamma = -\int_\Omega \frac{\partial \phi}{\partial t}\phi^* d\Omega \tag{18}$$

DUAL RECIPROCITY APPROACH

In order to obtain a boundary integral which is equivalent to the domain integral in (17) and (18), a dual reciprocity approximation is introduced[12]. The basic idea is to expand the time-derivative $\partial\phi/\partial t$ in the form,

$$\dot{\phi} = \sum_{k=1}^{P} f_k(x, y)\alpha_k(t) \tag{19}$$

where the dot denotes temporal derivative. The above series involves a set of known functions f_k which are dependent only on geometry, and a set of unknown coefficients α_k which are time-dependent only. With this approximation, the domain integral becomes

$$\int_\Omega \dot{\phi}\phi^* d\Omega = \sum_{k=1}^{P} \alpha_k \int_\Omega f_k \phi^* d\Omega \tag{20}$$

The next step is to consider that, for each function f_k, there exists a related function ψ_k which is a particular solution of the equation:

$$D\nabla^2\psi - v_x\frac{\partial \psi}{\partial x} - v_y\frac{\partial \psi}{\partial y} - k\psi = f \tag{21}$$

Thus, the domain integral can be recast in the form:

$$\int_\Omega \dot{\phi}\phi^* \, d\Omega = \sum_{k=1}^{P} \alpha_k \int_\Omega \left(D\nabla^2\psi_k - v_x \frac{\partial \psi_k}{\partial x} - v_y \frac{\partial \psi_k}{\partial y} - k\psi_k \right) \phi^* \, d\Omega \tag{22}$$

Substituting expansion (22) into (18), and applying integration by parts to the right side of the resulting equation, one finally arrives at a boundary integral equation of the form

$$c(\xi)\phi(\xi) - D\int_\Gamma \phi^* \frac{\partial \phi}{\partial n} d\Gamma + D\int_\Gamma \phi \frac{\partial \phi^*}{\partial n} d\Gamma + \int_\Gamma \phi\phi^* v_n \, d\Gamma$$
$$= \sum_{k=1}^{P} \alpha_k \left[c(\xi)\psi_k(\xi) - D\int_\Gamma \phi^* \frac{\partial \psi_k}{\partial n} d\Gamma + D\int_\Gamma \psi_k \frac{\partial \phi^*}{\partial n} d\Gamma + \int_\Gamma \psi_k \phi^* v_n \, d\Gamma \right] \tag{23}$$

NUMERICAL SOLUTION

For the numerical solution of the problem, (23) is discretized in a similar way to (9), i.e.

$$c_i\phi_i - D\sum_{j=1}^{N} \int_{\Gamma_j} \phi^* \frac{\partial \phi}{\partial n} d\Gamma + D\sum_{j=1}^{N} \int_{\Gamma_j} \left(\frac{\partial \phi^*}{\partial n} + \frac{v_n}{D}\phi^* \right) \phi \, d\Gamma$$
$$= \sum_{k=1}^{P} \alpha_k \left[c_i\psi_{ik} - D\sum_{j=1}^{N} \int_{\Gamma_j} \phi^* \frac{\partial \psi_k}{\partial n} d\Gamma + D\sum_{j=1}^{N} \int_{\Gamma_j} \left(\frac{\partial \phi^*}{\partial n} + \frac{v_n}{D}\phi^* \right) \psi_k \, d\Gamma \right] \tag{24}$$

Next, the variation of functions ϕ, $q = \partial\phi/\partial n$, ψ and $\eta = \partial\psi/\partial n$ within each boundary element is approximated by interpolating from the values at the element nodes. It should be noted that functions ψ and η need not be approximated as they are known functions for a specified set f. However, doing so will greatly improve the computer efficiency of the technique with only a minor sacrifice in accuracy.

Applying (24) to all boundary nodes, taking into account the previous interpolations, results in the following system of equations (see expression 13):

$$\mathbf{H}\boldsymbol{\phi} - \mathbf{G}\mathbf{q} = (\mathbf{H}\boldsymbol{\psi} - \mathbf{G}\boldsymbol{\eta})\boldsymbol{\alpha} \tag{25}$$

In the above system, the same matrices **H** and **G** as for steady problems are used on both sides. Matrices $\boldsymbol{\psi}$ and $\boldsymbol{\eta}$ are also geometry-dependent square matrices (assuming, for simplicity, that the number of terms in expansion (19) is equal to the number of boundary nodes), and $\boldsymbol{\phi}$, **q** and $\boldsymbol{\alpha}$ are vectors of nodal values.

By applying expression (19) at all boundary nodes and inverting, one arrives at:

$$\boldsymbol{\alpha} = \mathbf{F}^{-1}\dot{\boldsymbol{\phi}} \tag{26}$$

which, substituted into (25) results in:

$$\mathbf{C}\dot{\boldsymbol{\phi}} + \mathbf{H}\boldsymbol{\phi} = \mathbf{G}\mathbf{q} \tag{27}$$

with

$$\mathbf{C} = -(\mathbf{H}\boldsymbol{\psi} - \mathbf{G}\boldsymbol{\eta})\mathbf{F}^{-1}$$

System (27) can be integrated in time using standard time-stepping procedures. It should be stressed that the coefficients of matrices **H**, **G** and **C** all depend on geometry only, thus they can be computed once and stored.

Employing a general two-level time integration scheme for solution of (27), the following discrete form is obtained:

$$\left(\frac{1}{\Delta t}\mathbf{C} + \theta\mathbf{H} \right) \boldsymbol{\phi}^{m+1} - \theta\mathbf{G}\mathbf{q}^{m+1} = \left[\frac{1}{\Delta t}\mathbf{C} - (1-\theta)\mathbf{H} \right] \boldsymbol{\phi}^m + (1-\theta)\mathbf{G}\mathbf{q}^m \tag{28}$$

where θ is a parameter which positions the values of ϕ and $\mathbf{q}$ between time levels m and $m+1$. The right side of (28) is known at all times. Upon introducing the boundary conditions at time $(m+1)\Delta t$ the left side of the equation can be rearranged and the resulting system solved by using a standard direct procedure like Gauss elimination.

CHOICE OF FUNCTION f

Previous works on dual reciprocity schemes have shown that although a variety of functions can in principle be used as a basis for the approximation of ϕ, best results are normally obtained with simple expansions, the most popular of which is $f=1+r$, where r is the distance between pre-specified fixed points (poles) and the boundary nodes[12]. This choice is based on practical experience rather than formal mathematical analysis.

In the present work, it was decided to start with a simple form of particular solution ψ and find the corresponding expression for f by direct substitution into (21). The resulting expressions are:

$$\psi = r^3$$

$$\eta = 3r[(x-x_k)n_x + (y-y_k)n_y]$$

$$f = 9Dr - 3r[(x-x_k)v_x + (y-y_k)v_y] - kr^3$$

in which (x_k, y_k) and (x, y) are the coordinates of the kth pole and a general point, respectively.

The above choice was motivated by a previous successful experience with axisymmetric diffusion problems in which a similar approach was used[16]. It is interesting to notice that the set of functions f produced depends not only on r but also on the diffusivity, velocity components and the reaction rate, thus it will behave differently when diffusion or convection is the dominating process.

APPLICATIONS

The dual reciprocity boundary element formulation developed in this work was initially applied to the moving bar problem previously analysed in a steady state, assuming the initial condition $\phi_0=0$. The geometrical and physical parameters are the same as before, and the discretization again that of *Figure 1* with an extra internal pole at the centre of the rectangular region. The discontinuity of the normal flux at corners was considered by allowing corner nodes to have 3 degrees of freedom, i.e. ϕ, $\partial\phi/\partial n$ before the corner and $\partial\phi/\partial n$ after the corner, and prescribing 2 of these values. It is noted that the use of double nodes is not permitted with the dual reciprocity scheme because it leads to equal rows and columns in matrix **F**, rendering it singular and non-invertible.

Results for the variation of ϕ along x, at several time levels, are presented in *Figures 8* to *10* for the velocity values $v_x=0.2$, 1.0 and 6.0, respectively, compared to analytical solutions[17]. The BEM results were obtained using $\theta=1$ in (28) and a variable time step. The starting value of Δt was 0.10 for $v_x=0.2$, 0.025 for $v_x=1.0$ and 0.01 for $v_x=6.0$, but these values were increased during the analysis. It can be seen from the graphs that the accuracy of the dual reciprocity boundary element formulation is very good in all cases, with no oscillations or damping of the wave front.

Next, a similar problem was studied with the downstream boundary condition $\partial\phi/\partial n=0$ at $x=L$ and introducing first-order reaction. Once more, the discretization of *Figure 1* was employed, with $\theta=1$ and $\Delta t=0.05$. *Figures 11* to *16* present results for different values of v_x and k, i.e. $v_x=1$ or 6 and $k=0$, 0.278 or 1.389. The agreement with the analytical solutions given by van Genuchten and Alves[18] is again excellent.

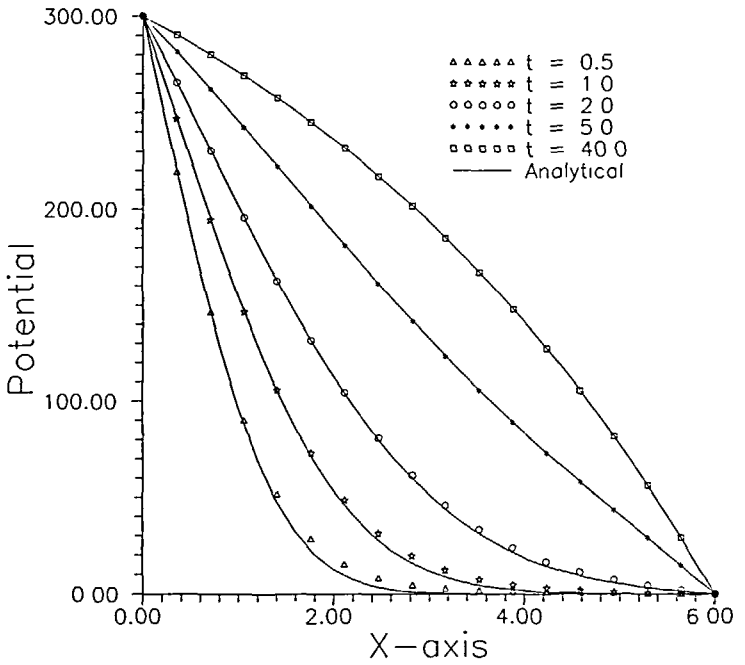

Figure 8 Results for $v_x=0.2$

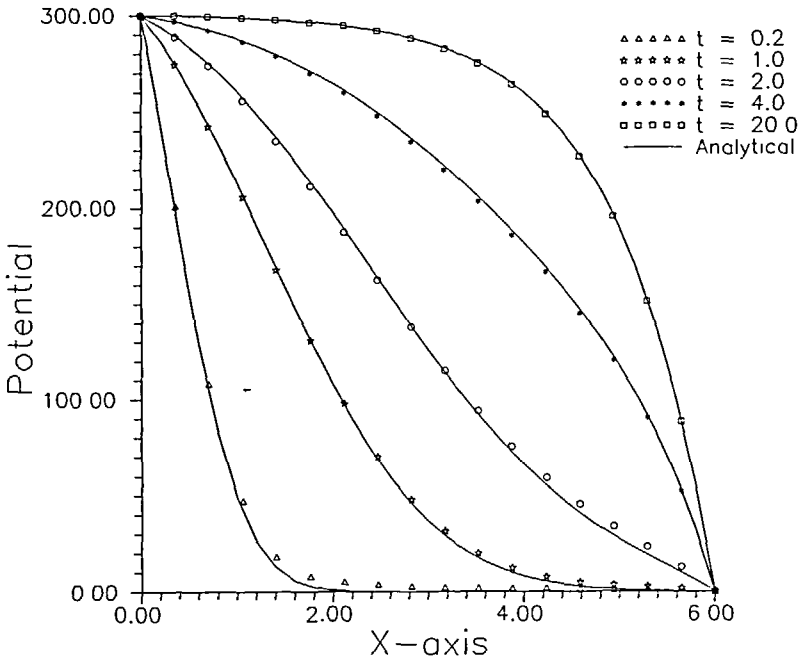

Figure 9 Results for $v_x=1$

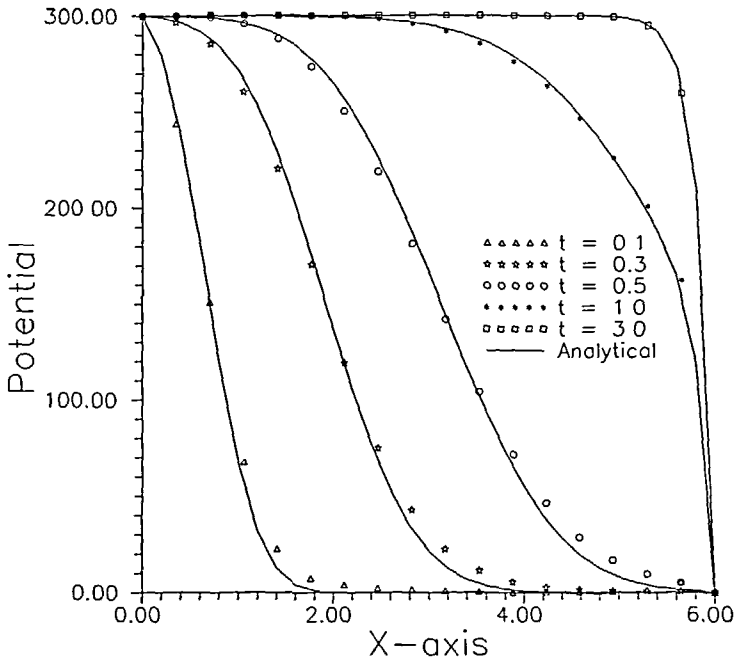

Figure 10 Results for $v_x = 6$

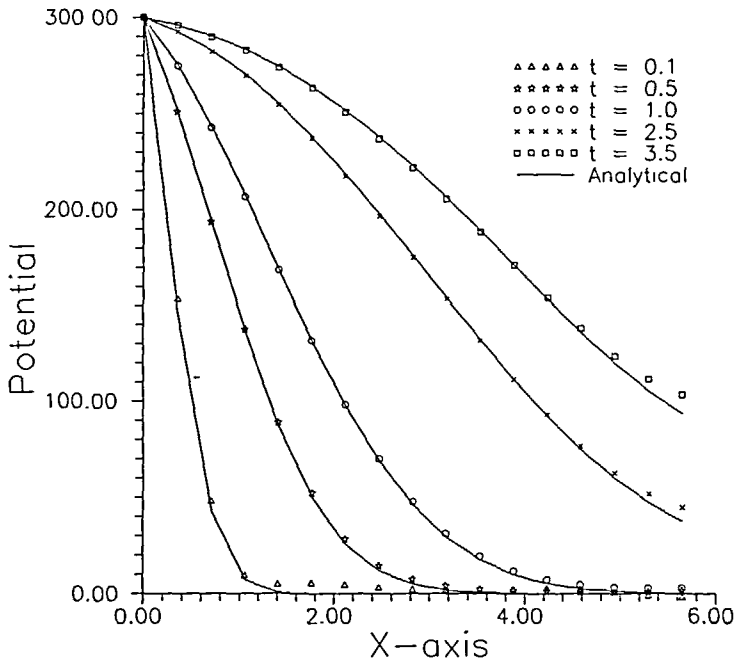

Figure 11 Results for $v_x = 1$, $k = 0$

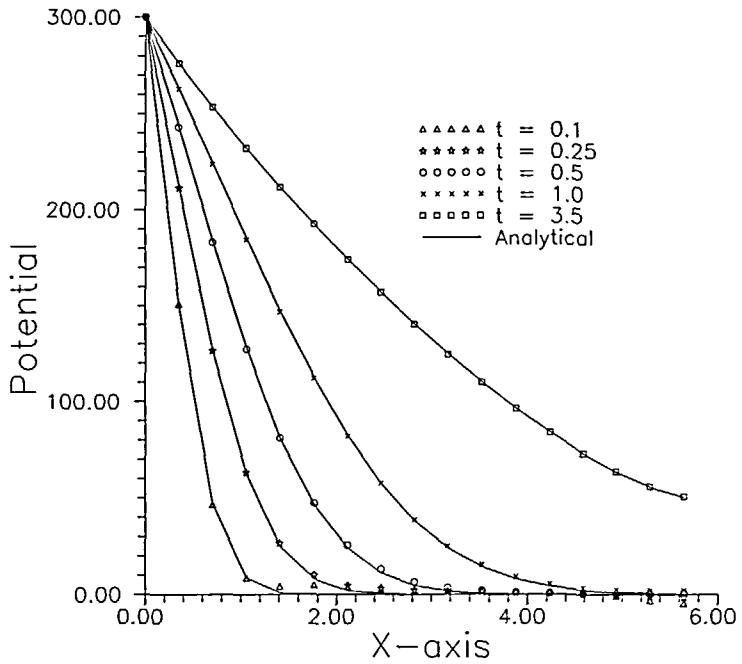

Figure 12 Results for $v_x = 1$, $k = 0.278$

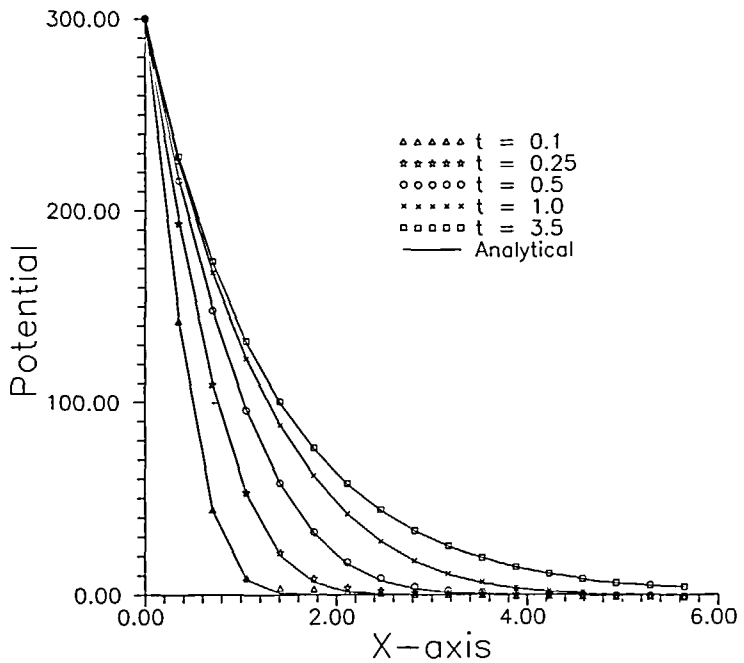

Figure 13 Results for $v_x = 1$, $k = 1.389$

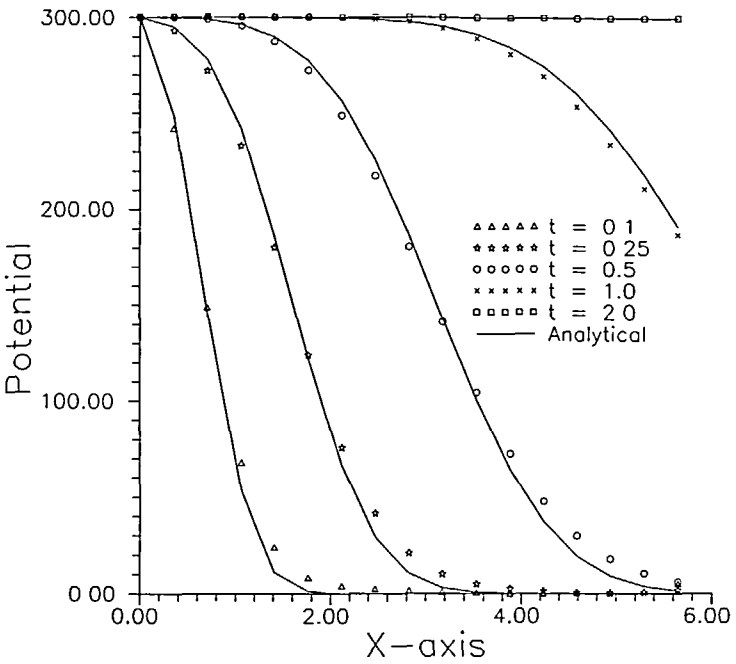

Figure 14 Results for $v_x=6$, $k=0$

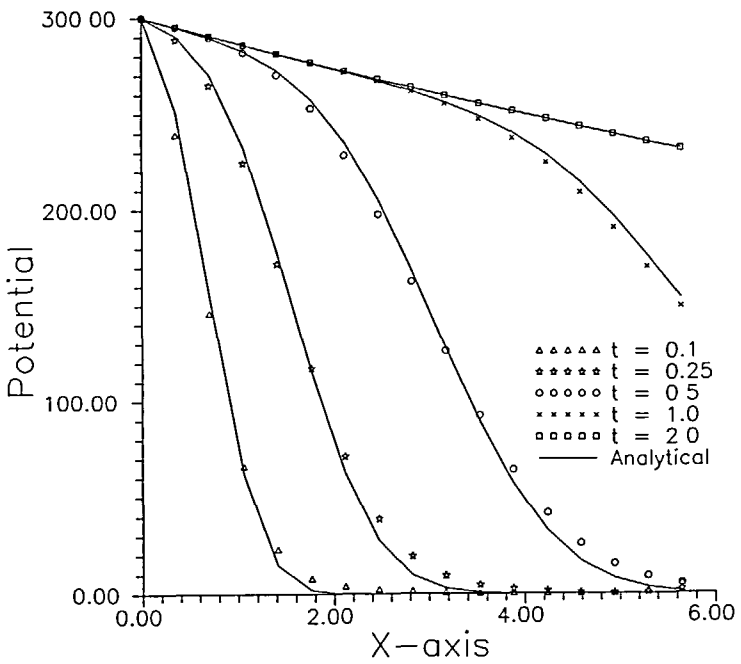

Figure 15 Results for $v_x=6$, $k=0.278$

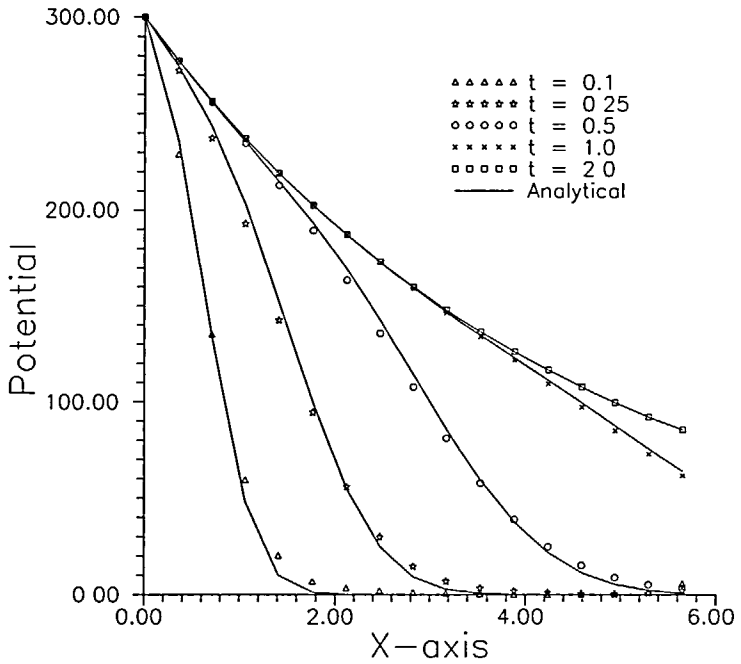

Figure 16 Results for $v_x=6$, $k=1.389$

CONCLUSIONS

This paper presented a formulation of the boundary element method for two-dimensional transient convection–diffusion problems, employing the fundamental solution of the corresponding steady-state equation and a dual reciprocity approximation of the time-derivative of ϕ. Results of applications have shown that the solutions do not display numerical problems of oscillations and damping of the wave front, common in finite difference and finite element formulations. It should be stressed that the formulation can be extended to variable velocity fields by splitting these into a constant part and a perturbation, and applying a similar dual reciprocity approximation to the perturbation[19].

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